Nonconvex compressive sensing
Fast, easy, and better

Rick Chartrand
Los Alamos National Laboratory

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In theory, there’s no difference between theory and practice. In practice, there is.

–Yogi Berra
Outline

Examples (Better)

A nonconvex objective for fast minimization (Fast, Easy)

High-dimensional data modeling

Avoiding local minima (Easy)

Summary
Motivating example

Suppose we want to reconstruct an image from samples of its Fourier transform. How many samples do we need?

Consider radial sampling, such as in MRI or (roughly) CT.

Shepp-Logan phantom, $s$
Nonconvexity is better

Fewer measurements are needed with nonconvex minimization:

$$\min_x H(\nabla x), \text{ subject to } (F x)|_\Omega = (F s)|_\Omega.$$
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With $H(x) = ||x||_1$, solution is $x = s$ with 18 lines ($\frac{\Omega}{x} = 6.9\%$).

backprojection, 18 lines  \quad \ell^1, 18 lines
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With $H(x) = \|x\|_1$, solution is $x = s$ with 18 lines ($\frac{\Omega}{|x|} = 6.9\%$).

With $H$ a nonconvex functional to be described shortly, 9 lines suffice ($\frac{\Omega}{|x|} = 3.5\%$). (More than $10^{4500}$ local minima!)

backprojection, 18 lines
\[ \ell^1, 18 \text{ lines} \]
nonconvex, 9 lines
\[ \ell^1, 9 \text{ lines} \]
3-D tomography

Six radiographs allow reconstruction of a stalagmite segment:

- Radiograph
- Isosurface
- Slice

(with Gary Sandine, LANL)
Frequency extrapolation

Consider the task of reconstructing an image with small test objects, from a $512 \times 512$ grid of samples of its continuous Fourier transform:

- high-res. phantom
- zoom-in on test objects
- Inverse DFT of the data
- zoom-in

with Emil Sidky, U. Chicago/Radiology
Frequency extrapolation

IDFT of zero-padded data

zoom

CS reconstruction, convex

zoom

nonconvex

zoom
Frequency extrapolation

The Fourier transform of the reconstructions (scaled by $|\nu|^{3/2}$) shows that the nonconvex method results in better extrapolation.

4096 × 4096 FT data of object  FT of convex result  FT of nonconvex result
Application: very noisy data

Independent Gaussian noise of $\sigma = 1000$ is added to the real and imaginary parts of the DFT of the $2048 \times 2048$ Shepp-Logan phantom. We exploit the greater SNR of the low-frequency portion of the data.

- IFFT, SNR $-10.2$ dB (or $\sigma = 0.69$)
- IFFT of zero-padded $256 \times 256$ data, SNR $7.1$ dB
- Nonconvex reconstruction of $256 \times 256$ data, SNR $17.5$ dB
- Nonconvex reconstruction from $20\%$ of $256 \times 256$ data, SNR $14.2$ dB
Interferometric imaging

Given a network of $N$ telescopes, the correlation between the electric field at each pair gives us $\binom{N}{2}$ Fourier samples.
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Examples (Better)

A nonconvex objective for fast minimization (Fast, Easy)

High-dimensional data modeling

Avoiding local minima (Easy)

Summary
Many efficient algorithms rely on shrinkage (or soft thresholding). The solution of the problem

\[ \min_w \| w \|_1 + \frac{1}{2\lambda} \| x - w \|_2^2 \]

is given componentwise by:

\[ w_i = \frac{x_i}{|x_i|} \max\{0, |x_i| - \lambda\} \].
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Using $p$-shrinkage with $p < 1$ instead promotes sparsity more strongly:

$$w_i = \frac{x_i}{|x_i|} \max \{0, |x_i| - \lambda |x_i|^{p-1}\}.$$

What problem does this solve?
A new objective function

We can construct a function $G_p$ so that $p$-shrinkage solves the analogous problem:

$$\min_w G_p(w) + \frac{1}{2\lambda} \| x - w \|_2^2$$
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For $p \leq 1$, $g$ is radial, radially strictly increasing, nonnegative, nonsmooth (at 0), continuous, and satisfies the triangle inequality. For large $x$, $g_p(x)$ grows like $x^p / p$. 
Our nonconvex generalization of split Bregman (or ADMM) is fast, and readily parallelizable. For example, to solve:

$$\min_x G_p(x), \text{ subject to } (\mathcal{F}\Phi x)|_\Omega = b$$

where the dictionary $\Phi$ gives a sparse representation of our signal, we iterate:
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where the dictionary \( \Phi \) gives a sparse representation of our signal, we iterate:

1. a \( p \)-shrinkage,
2. solving a linear system with an explicit, fast inverse, and
3. updating Lagrange multipliers.
function x = splitFourierterse( b, M, Phi, PhiT, mu, lambda, p, ep, iter )
[ m, n ] = size( b );
w = zeros( m, n );
Lam1 = zeros( m, n );
Lam2 = zeros( m, n );
% main loop
for ii = 1 : iter
    % solve for x in the Fourier domain
    rhs = ( w + Lam1 ) / lambda + mu * PhiT( n * ifft2( M .* ( b + Lam2 ) ) );
x = systeminverse( lambda, mu, M, Phi, PhiT, rhs );
    % update w
    w = shrink( x - Lam1, lambda, p, ep );
    % update Lagrange multipliers, using "method of multipliers"
    Lam1 = Lam1 + w - x;
    Lam2 = Lam2 + b - M .* fft2( Phi( x ) ) / n;
end
function x = systeminverse( lambda, mu, M, Phi, PhiT, y )
gmma = lambda^2 * mu / ( 1 + lambda * mu );
x = lambda * y - gmma * PhiT( ifft2( M .* fft2( Phi( y ) ) ) );
function y = shrink( x, lambda, p, ep )
% p-shrinkage using mollification
ax = sqrt( x .* conj( x ) );
y = max( ax - lambda * ( ax.^2 + ep ).^( p / 2 - 0.5 ), 0 );
id = ax ~= 0;
y( id ) = y( id ) .* x( id ) ./ ax( id );
Phantom example

We reconstruct an image from samples of its Fourier transform:

$$\min_x G_p(\nabla x), \text{ subject to } (\mathcal{F}x)|_\Omega = (\mathcal{F}s)|_\Omega.$$
A nonconvex objective for fast minimization (Fast, Easy)

**Phantom example**

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\]

- **Test image** \(s\)
- 9 lines/3.5% sampled, 21 seconds
- 10 lines/3.8% sampled, 5 seconds
Examples (Better)

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High-dimensional data modeling

Avoiding local minima (Easy)

Summary
Robust data modeling

We turn the task of modeling a high-dimensional dataset into a matrix optimization problem, by forming a matrix $D$ having each member of the dataset as a column.
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We turn the task of modeling a high-dimensional dataset into a matrix optimization problem, by forming a matrix $D$ having each member of the dataset as a column.

We seek to decompose $D$ into a sum $L + S$, where $L$ has low rank, and $S$ is sparse.

\[ D = L + S \]
Low rank + sparse decomposition

We could obtain our decomposition by solving the following:

$$\min_{L, S} \text{rank}(L) + \mu \|S\|_0, \text{ subject to } L + S = D.$$
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We obtain a robust, low-dimensional description of the dataset, and a set of salient features. We now examine the example of video, where each frame is a column of $D$. 
video $D$, 240 × 320 pixels, 288 frames
Video example

sparse component $S$
Video example

low-rank component $L$
Avoiding local minima (Easy)

Why might global minimization be possible?

Consider an $\epsilon$-regularized $\ell^p$ objective, restricted to the feasible plane:

$$\sum_{i=1}^{N} \left( x_i^2 + \epsilon \right)^{p/2}.$$ 

A moderate $\epsilon$ fills in the local minima.
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$\epsilon = 0.01$
Avoiding local minima (Easy)

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$\epsilon = 0.001$
Compressive sensing allows images and signals to be recovered from many fewer measurements than previously thought possible.

Nonconvex compressive sensing requires still fewer measurements.

State-of-the-art convex optimization methods can be extended to the nonconvex case, giving excellent computational efficiency.

Related matrix decomposition methods can extract interesting features from data without explicit modeling.

New applications continue to emerge.

math.lanl.gov/~rick